

Between 2- and 3-colorability

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Abstract

We consider the question of the existence of homomorphisms between $G_{n,p}$ and odd cycles when $p = c/n$, $1 < c \leq 4$. We show that for any positive integer ℓ , there exists $\varepsilon = \varepsilon(\ell)$ such that if $c = 1 + \varepsilon$ then w.h.p. $G_{n,p}$ has a homomorphism from $G_{n,p}$ to $C_{2\ell+1}$ so long as its odd-girth is at least $2\ell + 1$. On the other hand, we show that if $c = 4$ then w.h.p. there is no homomorphism from $G_{n,p}$ to C_5 . Note that in our range of interest, $\chi(G_{n,p}) = 3$ w.h.p., implying that there is a homomorphism from $G_{n,p}$ to C_3 .

1 Introduction

The determination of the chromatic number of $G_{n,p}$, where $p = \frac{c}{n}$ for constant c , is a central topic in the theory of random graphs. For $0 < c < 1$, such graphs contain, in expectation, a bounded number of cycles, and are almost-surely 3-colorable. The chromatic number of such a graph may be 2 or 3 with positive probability, according as to whether or not any odd cycles appear.

For $c \geq 1$, we find that the chromatic number $\chi(G_{n,\frac{c}{n}}) \geq 3$ with high probability, and letting $c_k := \sup_c \chi(G_{n,\frac{c}{n}}) \leq k$, it is known for all k and $c \in (c_k, c_{k+1})$ that $\chi(G_{n,\frac{c}{n}}) \in \{k, k+1\}$, see Łuczak [7] and Achlioptas and Naor [2]; for $k > 2$, the chromatic number may well be concentrated on the single value k , see Friedgut [5] and Achlioptas and Friedgut [1].

In this paper, we consider finer notions of colorability for the graphs $G_{n,\frac{c}{n}}$ for $c \in (1, c_3)$, by considering homomorphisms from $G_{n,\frac{c}{n}}$ to odd cycles $C_{2\ell+1}$. A homomorphism from a

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graph G to $C_{2\ell+1}$ implies a homomorphism to C_{2k+1} for $k < \ell$. As the 3-colorability of a graph G corresponds to the existence of a homomorphism from G to K_3 , the existence of a homomorphism to $C_{2\ell+1}$ implies 3-colorability. Thus considering homomorphisms to odd cycles $C_{2\ell+1}$ gives a hierarchy of 3-colorable graphs amenable to increasingly stronger constraint satisfaction problems. Note that a fixed graph having a homomorphism to any odd-cycle is bipartite.

Our main result is the following:

Theorem 1. *For any $\ell > 1$, there is an $\varepsilon > 0$ such that with high probability, $G_{n, \frac{1+\varepsilon}{n}}$ either has odd-girth $< 2\ell + 1$ or has a homomorphism to $C_{2\ell+1}$.*

Conversely, we expect the following:

Conjecture 1. *For any $c > 1$, there is an ℓ_c such that with high probability, there is no homomorphism from $G_{n, \frac{c}{n}}$ to $C_{2\ell+1}$ for $\ell \geq \ell_c$.*

As c_3 is known to be at least 4.03, the following confirms Conjecture 1 for a significant portion of the interval $(1, c_3)$.

Theorem 2. *For any $c > 2.774$, there is an ℓ_c such that with high probability, there is no homomorphism from $G_{n, \frac{c}{n}}$ to $C_{2\ell+1}$ for $\ell \geq \ell_c$.*

We also have that $\ell_4 = 2$:

Theorem 3. *With high probability, $G_{n, \frac{4}{n}}$ has no homomorphism to C_5 .*

Note that as $c_3 > 4.03 > 4$, we see that there are triangle-free 3-colorable random graphs without homomorphisms to C_5 . Our proof of Theorem 3 involves computer assisted numerical computations. The same calculations which rigorously demonstrate that $\ell_4 = 2$ suggest actually that $\ell_{3.75} = 2$ as well.

Our results can be reformulated in terms of the *circular chromatic number* of a random graph. Recall that the circular chromatic number $\chi_c(G)$ of G is the infimum r of circumferences of circles C for which there is an assignment of open unit intervals of C to the vertices of G such that adjacent vertices are assigned disjoint intervals. (Note that if circles C of circumference r were replaced in this definition with line segments S of length r , then this would give the ordinary chromatic number $\chi(G)$.) It is known that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, that $\chi_c(G)$ is always rational, and moreover, that $\chi_c(G) \leq \frac{p}{q}$ if and only if G has a homomorphism to the circulant graph $C_{p,q}$ with vertex set $\{0, 1, \dots, q-1\}$, with $v \sim u$ whenever $\text{dist}(v, u) := \min\{|v-u|, v+q-u, u+q-v\} \geq q$. (See [9].) Since $C_{2\ell+1, \ell}$ is the odd cycle $C_{2\ell+1}$ our results can be restated as follows:

Theorem 4. *In the following, inequalities for the circular chromatic number hold with high probability.*

1. For any $\delta > 0$, there is an $\varepsilon > 0$ such that, $G = G_{n, \frac{1+\varepsilon}{n}}$ has $\chi_c(G) \leq 2 + \delta$ unless it has odd girth $\leq \frac{2}{\delta}$.
2. For any $c > 2.774$, there exists $r > 2$ such that $\chi_c(G_{n, \frac{c}{n}}) > r$.
3. $2.5 \leq \chi_c(G_{n, \frac{4}{n}}) < 3$.

Note that for any c and $\ell > 1$, there is positive probability that $G_{n, \frac{c}{n}}$ has odd girth $< 2\ell + 1$, and a positive probability that it does not. In particular, as the probability that $G_{n, \frac{c}{n}}$ has small odd-girth can be computed precisely, Theorem 1 gives an exact probability in $(0, 1)$ that $G_{n, \frac{1+\varepsilon}{n}}$ has a homomorphism to $C_{2\ell+1}$. Indeed, Theorem 1 implies that if $c = 1 + \varepsilon$ and ε is sufficiently small relative to ℓ , then

$$\lim_{n \rightarrow \infty} \Pr(\chi_c(G_{n, \frac{c}{n}}) \in (2 + \frac{1}{\ell+1}, 2 + \frac{1}{\ell}]) = e^{-\phi_\ell(c)} - e^{-\phi_{\ell+1}(c)}, \quad (1)$$

where

$$\phi_\ell(c) = \sum_{i=1}^{\ell-1} \frac{c^{2i+1}}{2(2i+1)}.$$

We close with two more conjectures. The first concerns a sort of pseudo-threshold for having a homomorphism to $C_{2\ell+1}$:

Conjecture 2. *For any ℓ , there is a $c_\ell > 1$ such that $G_{n, \frac{c}{n}}$ has no homomorphism to $C_{2\ell+1}$ for $c > c_\ell$, and has either odd-girth $< 2\ell + 1$ or has a homomorphism to $C_{2\ell+1}$ for $c < c_\ell$.*

The second asserts that the circular chromatic numbers of random graphs should be dense.

Conjecture 3. *There are no real numbers $2 \leq a < b$ with the property that for any value of c , $\Pr(\chi_c(G_{n, \frac{c}{n}}) \in (a, b)) \rightarrow 0$.*

Note that our Theorem 1 confirms this conjecture for the case $a = 2$.

2 Structure of the paper

We prove Theorem 1 in Section 3. We first prove some structural lemmas and then we show, given the properties in these lemmas, that we can algorithmically find a homomorphism. We prove Theorem 2 in Section 4 by the use of a simple first moment argument. We prove Theorem 3 in Section 5. This is again a first moment calculation, but it has required numerical assistance in its proof.

3 Finding homomorphisms

Lemma 1. *If $\alpha < 1/10$ and c is a positive constant where*

$$c < c_0 = \exp \left\{ \frac{1 - 6\alpha}{3\alpha} \right\}$$

then w.h.p. any two cycles of length less than $\alpha \log n$ in $G_{n,p}$, $p = \frac{c}{n}$, are at distance more than $\alpha \log n$.

Proof If there are two cycles contradicting the above claim, then there exists a set S of size $s \leq 3\alpha \log n$ that contains at least $s + 1$ edges. The expected number of such sets can be bounded as follows:

$$\begin{aligned} \sum_{s=4}^{3\alpha \log n} \binom{n}{s} \binom{\binom{s}{2}}{s+1} \left(\frac{c}{n}\right)^{s+1} &\leq \sum_{s=4}^{3\alpha \log n} \left(\frac{ne}{s}\right)^s \left(\frac{se}{2}\right)^{s+1} \left(\frac{c}{n}\right)^{s+1} \\ &\leq \frac{3c\alpha \log n}{n} \sum_{s=4}^{3\alpha \log n} \left(\frac{ce^2}{2}\right)^s \\ &< \frac{(ce^2)^{3\alpha \log n} \log n}{n} \\ &= o(1). \end{aligned}$$

□

Our next lemma is concerned with cycles in K_2 which is the 2-core of $G_{n,p}$. The 2-core of a graph is the graph induced by the edges that are in at least one cycle. When $c > 1$, the 2-core consists of a linear size sub-graph together with a few vertex disjoint cycles. By few we mean that in expectation, there are $O(1)$ vertices on these cycles.

Let $0 < x < 1$ be such that $xe^{-x} = ce^{-c}$. Then w.h.p. K_2 has

$$\nu \sim (1-x) \left(1 - \frac{x}{c}\right) n \text{ vertices and } \mu \sim \left(1 - \frac{x}{c}\right)^2 \frac{cn}{2} \text{ edges.}$$

(See for example Pittel [8]).

If $c = 1 + \varepsilon$ for ε small and positive then $x = 1 - \eta$ where $\eta = \varepsilon + a_1 \varepsilon^2$, $|a_1| \leq 2$ for $\varepsilon < 1/10$.

The degree sequence of K_2 can be generated as follows, see for example Aronson, Frieze and Pittel [3]: Let λ be the solution to

$$\frac{\lambda(e^\lambda - 1)}{e^\lambda - 1 - \lambda} = \frac{2\mu}{\nu} \sim \frac{c-x}{1-x} = \frac{2+a_1\varepsilon}{1+a_1\varepsilon}.$$

We deduce from this that

$$\lambda \leq 4|a_1|\varepsilon \leq 8\varepsilon.$$

We generate the degrees $d(1), d(2), \dots, d(\nu)$ as independent copies of the random variable Z where for $d \geq 2$,

$$\Pr(Z = d) = \frac{\lambda^d}{d!(e^\lambda - 1 - \lambda)}.$$

We condition that the sum $D_1 = d(1) + d(2) + \dots + d(n) = 2\mu$. We let

$$\begin{aligned} \theta_k &= \frac{\Pr(d(i) = d_i, i = 1, 2, \dots, k \mid D_1 = 2\mu)}{\Pr(d(i) = d_i, i = 1, 2, \dots, k)} \\ &= \frac{\Pr(d(k+1) + \dots + d(n) = 2\mu - (d_1 + \dots + d_k))}{\Pr(d(1) + \dots + d(n) = 2\mu)}. \end{aligned}$$

It is shown in [3] that if Z_1, Z_2, \dots, Z_N are independent copies of Z then

$$\Pr(Z_1 + \dots + Z_N = N\mathbf{E}(Z) - t) = \frac{1}{\sigma\sqrt{2\pi N}} \left(1 + O\left(\frac{t^2 + 1}{N\sigma^2}\right) \right) \quad (2)$$

where $\sigma^2 = \Theta(1)$ is the variance of Z .

We observe next that the maximum degree in $G_{n,p}$ and hence in K_2 is q.s.¹ at most $\log n$. It follows from this and (2) that

$$\theta_k = 1 + o(1) \text{ for } k \leq \log^2 n \text{ and } \theta_k = O(n^{1/2}) \text{ in general.}$$

Lemma 2. *For any α, β , there exists $c_0 > 1$ such that w.h.p. any cycle of length greater than $\alpha \log n$ in the 2-core of $G_{n,p}$, $p = \frac{c}{n}$, $1 < c < c_0$, has at most $\beta \log n$ vertices of degree ≥ 3 .*

Proof Suppose that

$$e^{1+8\varepsilon} \left(\frac{8\varepsilon e}{\beta} \right)^\beta < 1.$$

We will show then that w.h.p. the K_2 does not contain a cycle C where (i) $|C| \geq \alpha \log n$ and (ii) C contains $\beta|C|$ vertices of degree greater than two.

We can bound the probability of the existence of a “bad” cycle C as follows: In the following display we choose the vertices of our cycle in $\binom{\nu}{k}$ ways and then arrange these vertices in a cycle C in $(k-1)!/2$ ways. Then we choose βk vertices to have degree at least three. We then sum over possible degree sequences for the vertices in C . This explains the factor $\theta_k \prod_{i=1}^k \frac{\lambda^{d_i}}{d_i!(e^\lambda - 1 - \lambda)}$. We now resort to using the configuration model of Bollobás [4]. This would explain the product $\prod_{i=1}^k \frac{d_i(d_i-1)}{2\mu-2i+1}$. We use the denominator $2\mu - k$ to simplify the calculation. The configuration model computation will inflate our estimate by a constant

¹A sequence of events \mathcal{E}_n is said to occur *quite surely* q.s. if $\Pr(\neg \mathcal{E}_n) = O(n^{-C})$ for any constant $C > 0$.

factor that we hide with the notation \leq_b . We write $A \leq_b B$ for $A = O(B)$ when $O(B)$ is “ugly looking”.

$$\begin{aligned}
\Pr(\exists C) &\leq_b \sum_{k=\alpha \log n}^{\nu} \binom{\nu}{k} \frac{(k-1)!}{2} \binom{k}{\beta k} \theta_k \sum_{\substack{d_1, \dots, d_{\beta k} \geq 3 \\ d_{\beta k+1}, \dots, d_k \geq 2}} \prod_{i=1}^k \left(\frac{\lambda^{d_i}}{d_i! (e^\lambda - 1 - \lambda)} \cdot \frac{d_i(d_i-1)}{2\mu - 2k} \right) \\
&\leq \sum_{k=\alpha \log n}^{\nu} \frac{1}{2k} \left(\frac{\nu}{(2\mu - 2k)(e^\lambda - 1 - \lambda)} \right)^k \lambda^{2k} \binom{k}{\beta k} \theta_k \sum_{\substack{d_1, \dots, d_{\beta k} \geq 3 \\ d_{\beta k+1}, \dots, d_k \geq 2}} \prod_{i=1}^k \frac{1}{(d_i - 2)!} \\
&\leq \sum_{k=\alpha \log n}^{\nu} \frac{e^{k^2/\mu}}{2k} \left(\frac{\nu}{2\mu(e^\lambda - 1 - \lambda)} \right)^k \lambda^{2k} \binom{k}{\beta k} \theta_k (e^\lambda - 1)^{\beta k} e^{(1-\beta)k\lambda} \\
&= \sum_{k=\alpha \log n}^{\nu} \frac{e^{k^2/\mu}}{2k} \left(\frac{\lambda}{e^\lambda - 1} \right)^k \binom{k}{\beta k} \theta_k (e^\lambda - 1)^{\beta k} e^{(1-\beta)k\lambda} \\
&\leq \sum_{k=\alpha \log n}^{\nu} \frac{\theta_k}{2k} \left(e^{k/\mu} \cdot \frac{\lambda}{(e^\lambda - 1)^{1-\beta}} \cdot \left(\frac{e}{\beta} \right)^\beta \cdot e^{(1-\beta)\lambda} \right)^k \\
&\leq \sum_{k=\alpha \log n}^{\nu} \frac{\theta_k}{2k} \left(e \cdot \lambda^\beta \cdot \left(\frac{e}{\beta} \right)^\beta \cdot e^\lambda \right)^k \\
&= o(1).
\end{aligned}$$

□

Lemma 3. *For any α and any $k \in \mathbb{N}$, there exists $\varepsilon_0 > 0$ such that w.h.p. we can decompose the edges of the $G = G_{n,p}$, $p = \frac{1+\varepsilon}{n}$, $0 < \varepsilon < \varepsilon_0$, as $F \cup M$, where F is a forest, and where the distance in F between any two edges in M is at least k .*

Proof By choosing $\beta < \frac{1}{2k}$ in Lemma 2 we can find, in every cycle of length $> \alpha \log n$ of the 2-core K_2 of G (which includes all cycles of G), a path of length at least $2k + 1$ whose interior vertices are all of degree 2. We can thus choose in each cycle of K_2 of length $> \alpha \log n$ such a path of maximum length, and let \mathcal{P} denote the set of such paths. (Note that, in general, there will be fewer paths in \mathcal{P} than long cycles in K_2 due to duplicates, but that the elements of \mathcal{P} are nevertheless disjoint paths in K_2 .) We now choose from each path in \mathcal{P} an edge from the center of the path to give a set M_1 . Note that the set of cycles in $G \setminus M_1$ is the same as the set of cycles in $G \setminus \bigcup_{P \in \mathcal{P}} P$. (In particular, the only cycles which remain have length $\leq \alpha \log n$ and are at distance $\geq k$ from M .) Thus, letting M_2 consist of one edge from each cycle of $G \setminus M_1$, Lemma 1 implies that $M = M_1 \cup M_2$ is as desired. □

Proof of Theorem 1. Our goal in this section is to give a $C_{2\ell+1}$ -coloring of $G = G_{n, \frac{1+\varepsilon}{n}}$ for $\varepsilon > 0$ sufficiently small. By this we will mean an assignment $c : V(G) \rightarrow \{0, 1, \dots, 2\ell\}$ such that $x \sim y$ in G implies that $c(x) \sim c(y)$ as vertices of $C_{2\ell+1}$; that is, that $x = y \pm 1 \pmod{2\ell+1}$.

Consider a decomposition of G as $F \cup M$ as given by Lemma 3, with $k = 4\ell - 2$.

We begin by 2-coloring F . Let $c_F : V \rightarrow \{0, 1\}$ be such a coloring. Our goal will be to modify this coloring to give a good $C_{2\ell+1}$ coloring of S .

Let \mathcal{B} be the set of edges $xy \in M$ for which $c_F(x) = c_F(y)$, and let B be a set of distinct representatives for \mathcal{B} , and for $i = 0, 1$, let $B^i = \{v \in B \mid c_F(v) = i\}$.

We now define a new $C_{2\ell+1}$ coloring $c : V \rightarrow \{0, 1, \dots, 2\ell\}$, by

$$c(v) = \begin{cases} c_F(v) & \text{if } \text{dist}_F(v, B) \geq 2\ell - 1 \\ c_F(x) - (-1)^j(\text{dist}_F(x, v) + 1) & \text{if } \exists x \in B^j \text{ s.t. } \text{dist}(x, v)_F < 2\ell - 1. \end{cases} \quad (3)$$

(Color addition and subtraction are computed modulo $2\ell + 1$.)

Since edges in M are separated by distances $\geq 4\ell - 2$, this coloring is well-defined (i.e., there is at most one choice for x). Moreover, c is certainly a good $C_{2\ell+1}$ -coloring of F . Thus if c is not a good $C_{2\ell+1}$ -coloring of S , it is bad along some edge $xy \in M$. But if such an edge was already properly colored in the 2-coloring c_F , it is still properly colored by c , since it has distance $\geq 4\ell - 2 \geq 2\ell - 1$ from other edges in M . On the other hand, if previously we had $c_F(x) = c_F(y) = i$, and WLOG $x \in B^i$, then the definition of $c(v)$ gives that we now have that $c(x) \in \{i - 1, i + 1\}$ (modulo $2\ell - 1$). Thus if c is not a good $C_{2\ell+1}$ -coloring of S , then there is an edge $xy \in M$ such that $x \in B^i$ and y 's color also changes in the coloring c ; but by the distance between edges in M , this can only happen if x and y are at F -distance $< 2\ell - 1$. Note also that $c_F(x) = c_F(y)$ implies that $\text{dist}_F(x, y)$ is even. Thus in this case, $F \cup \{xy\}$ contains an odd cycle of length $\leq 2\ell - 1$, and so G has odd girth $< 2\ell + 1$, as desired. \square

4 Avoiding homomorphisms to long odd cycles

For large ℓ , one can prove the non-existence of homomorphisms to $C_{2\ell+1}$ using the following simple observation:

Observation 4. *If G has a homomorphism to $C_{2\ell+1}$, then G has an induced bipartite subgraph with at least $\frac{2\ell}{2\ell+1}|V(G)|$ vertices.*

Proof. Delete the smallest color class. \square

Proof of Theorem 2. The probability that $G_{n, \frac{c}{n}}$ has an induced bipartite subgraph on βn vertices is at most

$$\binom{n}{\beta n} 2^{\beta n} \left(1 - \frac{c}{n}\right)^{\beta^2 n^2 / 4} < \left(\frac{2^\beta e^{-c\beta^2/4}}{\beta^\beta (1-\beta)^{1-\beta}}\right)^n \quad (4)$$

The expression inside the parentheses is unimodal in β for fixed c , and, for $c > 2.774$, is less than 1 for $\beta > .999971$. In particular, for $c > 2.774$, $G_{n, \frac{c}{n}}$ has no homomorphism to $C_{2\ell+1}$ for $2\ell + 1 \geq 1,427,583$. \square

5 Avoiding homomorphisms to C_5

A homomorphism of $G = G_{n,p}$, $p = \frac{c}{n}$ into C_5 induces a partition of $[n]$ into sets $V_i, i = 0, 1, \dots, 4$. This partition can be assumed to have the following properties:

P1 The sets $V_i, i = 0, 1, \dots, 4$ are all independent sets.

P2 There are no edges between V_i and $V_{i+2} \cup V_{i-2}$. Here addition and subtraction in an index are taken to be modulo 5.

P3 Every $v \in V_i, i = 1, 2, 3, 4$ has a neighbor in V_{i-1} .

P4 Every $v \in V_2$ has a neighbor in V_3 .

Hatami [6], Lemma 2.1 shows that we can assume **P1, P2, P3**. Given **P1, P2, P3**, if $v \in V_2$ has no neighbors in V_3 then we can move v from V_2 to V_0 and still have a homomorphism. Furthermore, this move does not upset **P1, P2, P3**.

We let $|V_i| = n_i$ for $i = 0, 1, \dots, 4$. For a fixed partition we then have

$$\Pr(\mathbf{P1} \wedge \mathbf{P2}) = (1-p)^S \text{ where } S = \binom{n}{2} - \sum_{i=0}^4 n_i n_{i+1}. \quad (5)$$

$$\Pr(\mathbf{P3} \mid \mathbf{P1} \wedge \mathbf{P2}) = \prod_{i=1}^4 (1 - (1-p)^{n_{i-1}})^{n_i}. \quad (6)$$

$$\Pr(\mathbf{P4} \mid \mathbf{P1} \wedge \mathbf{P2} \wedge \mathbf{P3}) \leq \left(1 - \left(1 - \frac{1}{n_2}\right)^{n_3} (1-p)^{n_3}\right)^{n_2} \quad (7)$$

Equations (5) and (6) are self evident, but we need to justify (7). Consider the bipartite subgraph Γ of $G_{n,p}$ induced by $V_2 \cup V_3$. **P3** tells us that each $v \in V_3$ has a neighbor in V_2 . Denote this event by \mathcal{A} . Suppose now that we choose a random mapping ϕ from V_3 to V_2 . We then create a bipartite graph Γ' with edge set $E_1 \cup E_2$. Here $E_1 = \{xy : x \in V_3, y = \phi(x)\}$ and E_2 is obtained by independently including each of the $n_2 n_3$ possible edges between V_2 and V_3 with probability p . We now claim that we can couple Γ, Γ' so that $\Gamma \subseteq \Gamma'$.

Event \mathcal{A} can be construed as follows: A vertex in $v \in V_3$ chooses B_v neighbors in V_2 where B_v is distributed as a binomial $\text{Bin}(n_2, p)$, conditioned to be at least one. The neighbors of v in V_2 will then be a random B_v subset of V_2 . We only have to prove then that if v chooses B'_v random neighbors in Γ' then B'_v stochastically dominates B_v . But B'_v is one plus $\text{Bin}(n_2 - 1, p)$ and domination is easy to confirm. We have $n_2 - 1$ instead of n_2 , since we do not wish to count the edge v to $\phi(v)$ twice.

We now write $n_i = \alpha_i n$ for $i = 0, \dots, 4$. We are particularly interested in the case where $c = 4$. Now (4) implies that $G_{n, \frac{4}{n}}$ has no induced bipartite subgraph of size βn for $\beta > 0.94$. Thus we may assume that $\alpha_i \geq 0.06$ for $i = 0, \dots, 4$. In which case we can write

$$\Pr(\mathbf{P1} \wedge \mathbf{P2} \wedge \mathbf{P3} \wedge \mathbf{P4}) \leq e^{o(n)} \times \exp \left\{ -c \left(\frac{1}{2} - \sum_{i=0}^4 \alpha_i \alpha_{i+1} \right) n \right\} \times \left(\prod_{i=1}^4 (1 - e^{-c\alpha_{i-1}})^{\alpha_i} \right)^n \times (1 - e^{-\alpha_3/\alpha_2} e^{-c\alpha_3})^{\alpha_2 n}.$$

The number of choices for V_0, \dots, V_4 with these sizes is

$$\binom{n}{n_0, n_1, n_2, n_3, n_4} = e^{o(n)} \times \left(\frac{1}{\prod_{i=0}^4 \alpha_i^{\alpha_i}} \right)^n \leq 5^n.$$

Putting $\alpha_4 = 1 - \alpha_0 - \alpha_1 - \alpha_2 - \alpha_3$ and

$$b = b(c, \alpha_0, \alpha_1, \alpha_2, \alpha_3) = \frac{1}{\alpha_0^{\alpha_0} \alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \alpha_3^{\alpha_3} \alpha_4^{\alpha_4}} e^{c(\alpha_0 \alpha_4 - \frac{1}{2})} (e^{c\alpha_0} - 1)^{\alpha_1} (e^{c\alpha_1} - 1)^{\alpha_2} (e^{c\alpha_2} - 1)^{\alpha_3} (e^{c\alpha_3} - 1)^{\alpha_4} (1 - e^{-\alpha_3/\alpha_2} e^{-c\alpha_3})^{\alpha_2},$$

we see that since there are $O(n^4)$ choices for n_0, \dots, n_4 we have

$$\Pr(\exists \text{ a homomorphism from } G_{n, \frac{4}{n}} \text{ to } C_5) \leq e^{o(n)} \left(\max_{\substack{\alpha_0 + \dots + \alpha_3 \leq 0.94 \\ \alpha_0, \dots, \alpha_3 \geq 0.06}} b(4, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \right)^n. \quad (8)$$

In the next section, we describe a numerical procedure for verifying that the maximum in (8) is less than 1. This will complete the proof of Theorem 3.

6 Bounding the function.

Our aim now is to bound the partial derivatives of $b(4.0, \alpha_0, \alpha_1, \alpha_2, \alpha_3)$, to translate numerical computations of the function on a grid to a rigorous upper bound.

Before doing this we verify that w.h.p. $G_{n, p=\frac{4}{n}}$ has no independent set S of size $s = 3n/5$ or more. Indeed,

$$\Pr(\exists S) \leq 2^n (1-p)^{\binom{s}{2}} \leq 2^n e^{-18n/25} e^{12/5} = o(1).$$

In the calculations below we will make use of the following bounds: They assume that $0.06 \leq \alpha_i \leq 0.6$ for $i \geq 0$.

$$\begin{aligned} \log(\alpha_i) &> -2.82; \quad -1.31 < \log(e^{4\alpha_i} - 1) < 2.31; \quad \frac{e^{4\alpha_i}}{e^{4\alpha_i} - 1} < 4.69 \\ \frac{1}{e^{4\alpha_i} - 1} &< 3.69; \quad \log(e^{\alpha_3/\alpha_2 + 4\alpha_3} - 1) > -0.91; \quad \frac{1 + 4\alpha_2}{e^{\alpha_3/\alpha_2} e^{4\alpha_3} - 1} < 8.40. \end{aligned}$$

We now use these estimates to bound the absolute values of the $\frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_i}$. Our target value for these is 30. We will be well within these bounds except for $i = 2$

Taking logarithms to differentiate with respect to α_0 , we find

$$\begin{aligned} \frac{\partial b}{\partial \alpha_0} &= b(c, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \times \\ &\quad \left(c \left(-\alpha_0 + \alpha_1 + \frac{\alpha_1}{e^{\alpha_0 c} - 1} + \alpha_4 \right) - \log(\alpha_0) + \log(\alpha_4) - \log(e^{\alpha_3 c} - 1) \right). \end{aligned} \quad (9)$$

In particular, for $c = 4$,

$$\begin{aligned} \frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_0} &\geq -4\alpha_0 + \log(\alpha_4) - \log(e^{4\alpha_3} - 1) > -2.4 - 2.82 - 2.31, \\ \frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_0} &\leq 4 \left(\alpha_1 + \frac{\alpha_1}{e^{\alpha_0 c} - 1} + \alpha_4 \right) - \log(\alpha_0) - \log(e^{4\alpha_3} - 1) < 4 \times 4.69 + 2.82 + 1.31. \end{aligned}$$

Similarly, we find

$$\begin{aligned} \frac{\partial b}{\partial \alpha_1} &= b(c, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \times \\ &\quad \left(c \left(-\alpha_0 + \alpha_2 + \frac{\alpha_2}{e^{\alpha_1 c} - 1} \right) - \log(\alpha_1) + \log(\alpha_4) + \log \left(\frac{e^{\alpha_0 c} - 1}{e^{\alpha_3 c} - 1} \right) \right), \end{aligned} \quad (10)$$

and so for $c = 4$,

$$\begin{aligned} \frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_1} &\geq -4\alpha_0 + \log(\alpha_4) + \log(e^{4\alpha_0} - 1) - \log(e^{4\alpha_3} - 1) > -2.4 - 2.82 - 3.62, \\ \frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_1} &\leq 4 \left(\alpha_2 + \frac{\alpha_2}{e^{4\alpha_1} - 1} \right) - \log(\alpha_1) - \log(e^{4\alpha_3} - 1) < 2.4 \times 4.69 + 2.82 + 1.31. \end{aligned}$$

We next find that

$$\begin{aligned} \frac{\partial b}{\partial \alpha_2} &= b(c, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \times \\ &\quad c \left(-\alpha_0 + \alpha_3 + \frac{\alpha_3}{e^{\alpha_2 c} - 1} \right) - \frac{\alpha_3/\alpha_2}{e^{\alpha_3/\alpha_2 + c\alpha_3} - 1} + \\ &\quad \log \alpha_4 - \log \alpha_2 + \log(e^{\alpha_1 c} - 1) - \log(e^{\alpha_3 c} - 1) - \frac{\alpha_3}{\alpha_2} - c\alpha_3 - \log(e^{\alpha_3/\alpha_2 + c\alpha_3} - 1); \end{aligned} \quad (11)$$

and so for $c = 4$,

$$\frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_2} \geq -4\alpha_0 - \frac{\alpha_3}{\alpha_2} \frac{e^{\alpha_3/\alpha_2 + c\alpha_3}}{e^{\alpha_3/\alpha_2 + c\alpha_3} - 1} - \log(e^{\alpha_3/\alpha_2 + c\alpha_3} - 1) + \log(\alpha_4) + \log\left(\frac{e^{4\alpha_1} - 1}{e^{4\alpha_3} - 1}\right)$$

We need to be a little careful here. Now $\alpha_3/\alpha_2 \leq 10$ and if $\alpha_3/\alpha_2 \geq 9$ then $\alpha_3 \geq 0.54$ and then $\alpha_i \leq 0.46 - 3 \times .06 = 0.28$ for $i \neq 3$. We bound $-\frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_i}$ for both possibilities.

Continuing we get

$$\begin{aligned} \frac{\alpha_3}{\alpha_2} \geq 9 : \frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_2} &> -1.12 - 10.01 - 12.4 - 2.82 - 3.62 = -29.97, \\ \frac{\alpha_3}{\alpha_2} \leq 9 : \frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_2} &> -2.4 - 9.01 - 11.4 - 2.82 - 3.62, \\ \frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_2} &\leq 4 \left(\alpha_3 + \frac{\alpha_3}{e^{4\alpha_2} - 1} \right) - \log(\alpha_2) + \log\left(\frac{e^{4\alpha_1} - 1}{e^{4\alpha_3} - 1}\right) - \log(e^{\alpha_3/\alpha_2 + c\alpha_3} - 1) \\ &< 2.4 \times 3.69 + 2.82 + 3.62 + 0.91. \end{aligned}$$

Finally, we find that

$$\begin{aligned} \frac{\partial b}{\partial \alpha_3} &= b(c, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \times \\ &c \left(-\alpha_0 + \alpha_4 \frac{e^{c\alpha_3}}{e^{c\alpha_3} - 1} \right) + \frac{1 + c\alpha_2}{e^{\alpha_3/\alpha_2} e^{c\alpha_3} - 1} + \log(\alpha_4) - \log(\alpha_3) + \log\left(\frac{e^{\alpha_2 c} - 1}{e^{\alpha_3 c} - 1}\right) \quad (12) \end{aligned}$$

and so for $c = 4$

$$\begin{aligned} \frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_3} &\geq -4\alpha_0 + \log(\alpha_4) + \log(e^{4\alpha_2} - 1) - \log(e^{4\alpha_3} - 1) > -2.4 - 2.82 - 3.62, \\ \frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_3} &\leq 4\alpha_4 \frac{e^{4\alpha_3}}{e^{4\alpha_3} - 1} + \frac{1 + 4\alpha_2}{e^{\alpha_3/\alpha_2} e^{4\alpha_3} - 1} - \log(\alpha_3) + \log\left(\frac{e^{4\alpha_2} - 1}{e^{4\alpha_3} - 1}\right) \\ &< 2.4 \times 4.69 + 8.40 + 2.82 + 3.62. \end{aligned}$$

We see that $|\frac{1}{b} \cdot \frac{\partial b}{\partial \alpha_i}| < 30$ for all $0 \leq i \leq 3$. Thus, if we know that $b(c, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \leq B$ for some B , this means that we can bound $b(4, \alpha_0, \alpha_1, \alpha_2, \alpha_3) < \rho$ by checking that $b(4, \alpha_0, \alpha_1, \alpha_2, \alpha_3) < \rho - \varepsilon$ on a grid with step-size $\delta \leq \varepsilon/(2 \cdot B \cdot 30)$.

The C++ program in Appendix A checks that $b(4, \alpha_0, \alpha_1, \alpha_2, \alpha_3) < .949$ on a grid with step-size $\delta = .0008$ (it completes in around an hour or less on a standard desktop computer, and is available for download from the authors' websites). Suppose now that $B \geq 1$ is the supremum of $b(4, \alpha_0, \alpha_1, \alpha_2, \alpha_3)$ in the region of interest. For $\varepsilon = 60\delta B = 0.048B$, we must have at some δ -grid point that $b(4, \alpha_0, \alpha_1, \alpha_2, \alpha_3) \geq B - \varepsilon = .962B \geq .962$. This contradicts the computer-assisted bound of $< .949$ on the grid, completing the proof of Theorem 3. \square

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Xuding Zhu

A C++ code to check function bound

```
#include <iostream>
#include <math.h>
#include <stdlib.h>
using namespace std;
int main(int argc, char* argv[]){
    double delta=.0008;           //step size
    double maxIndSet=.6;           //no independent sets larger than this fraction
    double minClass=.06;           //all color classes larger than this fraction
    double val=0;
    double maxval=0;
    double maxa0,maxa1,maxa2,maxa3; //to record the coordinates of max value
    maxa0=maxa1=maxa2=maxa3=0;
    double A23,A,B,C;              //For precomputing parts of the function
    double c=4;
    for (double a3=minClass; a3 + 4*minClass<1; a3+=delta){
        B=exp(c*a3)-1;
        for (double a2=minClass; a3 + a2 + 3*minClass<1; a2+=delta){
            A23=1/(pow(a2,a2)*pow(a3,a3)) * exp(-c/2)
                * pow(exp(c*a2)-1,a3) * pow(1-exp(-a3/a2)*exp(-c*a3),a2);
            for (double a1=minClass;
                a3+a1<maxIndSet && a3 + a2 + a1 + 2*minClass<1;
                a1+=delta){
                A=A23/pow(a1,a1)* pow(exp(c*a1)-1,a2);
                for (double a0=max(max(minClass,.4-a2-a3),.4-a1-a3);
                    a2+a0<maxIndSet && a3+a0<maxIndSet
                        && a3 + a2 + a1 + a0 + minClass<1;
                    a0+=delta){
                    double a4=1-a0-a1-a2-a3;
                    C=exp(c*a0);
                    val=1/pow(a0,a0) * A * pow(B*C/a4,a4)* pow(C-1,a1);
                    if (val>maxval){
                        maxval=val;
                        maxa0=a0; maxa1=a1; maxa2=a2; maxa3=a3;
                    }
                }
            }
        }
    }
    cout << "Max is "<<maxval<<", obtained at ("
        <<maxa0<<","<<maxa1<<","<<maxa2<<","<<maxa3<<","
        <<1-maxa0-maxa1-maxa2-maxa3<<")"<<endl;
}
```

program output:

```
$/bound
```

```
Max is 0.948754, obtained at (0.2904,0.2568,0.1704,0.1632,0.1192)
```